



Stabilization of a Plate System with Nonlinear Boundary Damping and Delay

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Abstract

We study the qualitative properties of the solution of a plate system with boundary frictional damping and delay term having time-dependent coefficients $\alpha(t)$ and $\eta(t)$ and a variable exponent $m(x)$. Under suitable condition on the weight of the delayed feedback compared with the weight of the non-delayed feedback, we use the multiplier method and construct explicit energy decay estimates depending on both m and α , which extend and enhance earlier related results in the literature.

Keywords: plate equations, energy decay rates, frictional damping, delay.

2010 Mathematics Subject Classification: 35B40, 74D99, 93D15, 93D20

1. Introduction

In this paper we are concerned with the following problem

$$\left\{ \begin{array}{ll} u_{tt} + \Delta^2 u = 0, & \text{in } \Omega \times (0, \infty) \\ u = \frac{\partial u}{\partial \nu} = 0, & \text{on } \Gamma_0 \times (0, \infty) \\ \beta_1 u = 0, & \text{on } \Gamma_1 \times (0, \infty) \\ \beta_2 u = \alpha(t) |u_t|^{m(x)-2} u_t + \eta(t) |u_t(x, t - \tau)|^{m(x)-2} u_t(x, t - \tau), & \text{on } \Gamma_1 \times (0, \infty) \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & \text{in } \Omega \\ u_t(x, -t) = f(x, t), & \text{on } \Gamma_1 \times (0, \tau) \end{array} \right. \quad (1.1)$$

where Ω is a bounded domain of $\mathbb{R}^2$ with a smooth boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$, Γ_0 and Γ_1 are closed and disjoint, with $\text{meas}(\Gamma_0) > 0$, $\nu = (\nu_1, \nu_2)$ is the unit outward normal to $\partial\Omega$, $\eta = (-\nu_2, \nu_1)$ is the unit tangent positively oriented on $\partial\Omega$. This is a plate system with boundary frictional damping and delay term having time-dependent coefficients $\alpha(t)$ and $\eta(t)$ satisfying

$$\left\{ \begin{array}{l} \alpha : [0, \infty) \rightarrow (0, \infty) \text{ is a nonincreasing } C^1\text{-function and } \int_0^\infty \alpha(t) dt = \infty, \\ \eta : [0, \infty) \rightarrow \mathbb{R} \text{ is a } C^0\text{-function and } |\eta(t)| \leq q\alpha(t), \text{ where } 0 < q < 1, \end{array} \right. \quad (1.2)$$

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and a variable exponent $m \in C^1(\Gamma_1)$ satisfying

$$2 \leq m_1 \leq m(x) \leq m_2, \quad \text{where} \quad m_1 = \inf_{x \in \Gamma_1} m(x), \quad m_2 = \sup_{x \in \Gamma_1} m(x), \quad (1.3)$$

and the log-Hölder continuity condition

$$|m(x) - m(y)| \leq -\frac{A}{\log|x-y|}, \quad \text{for a.e. } x, y \in \Gamma_1 \text{ with } |x-y| < \delta, 0 < \delta < 1, A > 0. \quad (1.4)$$

The constant $\tau > 0$ represents the time delay, and f is the history function. We are denoting by β_1, β_2 the following differential operators:

$$\beta_1 u = \Delta u + (1 - \mu)B_1 u, \quad \beta_2 u = \frac{\partial \Delta u}{\partial \nu} + (1 - \mu) \frac{\partial B_2 u}{\partial \eta}$$

where

$$B_1 u = 2\nu_1 \nu_2 u_{x_1 x_2} - \nu_1^2 u_{x_2 x_2} - \nu_2^2 u_{x_1 x_1}, \quad B_2 u = (\nu_1^2 - \nu_2^2) u_{x_1 x_2} + \nu_1 \nu_2 (u_{x_2 x_2} - u_{x_1 x_1})$$

and $\mu \in (0, \frac{1}{2})$ represents the Poisson coefficient.

In the absence of delay, the control of the plate system by internal or boundary feedbacks has been widely studied in the literature. When the damping term behaves like $|u_t|^{m-2} u_t$ and m is a constant, several results about the asymptotic behavior of the solutions were established in [7,12,15,18,19], where it was shown that the energy decays exponentially when $m = 2$ and polynomially when $m > 2$. Recently, PDEs with variable exponents have attracted a lot of attention from researchers and academicians. This great interest is motivated by various physical applications, like the electro-rheological fluids, fluids with temperature-dependent viscosity, filtration processes in a porous media and image processing. Antontsev *et al.* [4] and Liao and Tan [21] looked at

$$u_{tt} + \Delta^2 u - \Delta u_t + |u_t|^{m(\cdot)-1} u_t = |u|^{p(\cdot)-2} u,$$

and showed that the solution is global with uniform energy decay rate if $m(\cdot) \geq p(\cdot)$, while a solution with negative initial energy and $m(\cdot) < p(\cdot)$ blows up in a finite time. More existence and blow up results of the wave equation were also given in [13,23,31]. The following problem

$$u_{tt} - \Delta u + \alpha(t)|u_t|^{m(\cdot)-1} u_t = 0,$$

with a time-dependent coefficient $\alpha(t)$, was investigated by Mustafa *et al.* [24,27] in bounded and unbounded domains and explicit energy decay rates were obtained. Messaoudi *et al.* [22] considered the system

$$\begin{cases} u_{tt} + \Delta^2 u + |u_t|^{m(\cdot)-1} u_t = f_1(u, v) \\ v_{tt} - \Delta v + |v_t|^{r(\cdot)-1} v_t = f_2(u, v) \end{cases}$$

and established a blow-up result as well as exponential and polynomial energy decay results depending on the source terms and the relations between the variable exponents.

The interaction between two types of damping, viscoelasticity and friction, was also a subject of interest studied by Al-Mahdi *et al.* [1,2,3], Gao and Gao [11], Hassan *et al.* [14], and Park and Kang [29]. Recently, Mustafa [25] studied a viscoelastic plate system having a boundary frictional damping with variable-exponent, and proved general energy decay results.

Time delays arise in many applications because most phenomena naturally depend not only on the present state but also on some past occurrences. It is well known that the delay can be a source of instability unless some additional control mechanisms are added, see [8]. Nicaise and Pignotti [28] studied the wave equation with linear damping and delay terms having constant coefficients a_0 and a , and determined that exponential stability or alternatively instability takes place when $a < a_0$ or $a \geq a_0$ respectively. The same scenario was

obtained by Badawi *et al.* [5] for Kirchhoff plate system with delay on the boundary control. These results were extended by Benaissa *et al.* [6] to the case of nonconstant time-dependent coefficients satisfying (1.2). In [10], Feng and Li treated with a plate equation having internal damping and delay terms of the nonlinear type $a|u|^{m-2}u$ where a and m are constants. Later, for wave equations, Kafini and messaoudi [16] covered the case when the exponent m is a variable function of x .

In the present work, we study system (1.1) including boundary damping and delay effects both with a variable exponent $m(x)$ satisfying (1.3)-(1.4) and modulated by time-dependent coefficients $\alpha(t)$ and $\eta(t)$ satisfying (1.2). We use the multiplier method and overcome the difficulty of having variable exponents to prove that the damping effect will dominate and lead to stability results supported by explicit formulae, depending on both m and α , for the energy decay rates which generalize earlier related results in the literature.

2. Preliminaries

Let $p : \Omega \rightarrow [1, \infty)$ be a measurable function, where Ω is a domain of $\mathbb{R}^n$. The Lebesgue and Sobolev spaces with a variable exponent $p(\cdot)$ are defined by

$$L^{p(\cdot)}(\Omega) = \{u : \Omega \rightarrow \mathbb{R}, \text{ measurable in } \Omega \text{ and } \int_{\Omega} |u(x)|^{p(x)} dx < \infty\}$$

and

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) : \nabla u \text{ exists and } |\nabla u| \in L^{p(\cdot)}(\Omega)\}.$$

For detailed properties of these spaces, we refer to [9,20]. Here, we only mention that the well-known Hölder's and Poincaré's inequalities and the embedding Theorems as well as the following trace Theorem are all applicable for these spaces.

• **Trace Theorem:** Let Ω be a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. Assume that $p \in C(\overline{\Omega})$ and $q \in C(\partial\Omega)$ such that

$$1 < p_1 \leq p(x) \leq p_2 < +\infty, \quad 1 < q_1 \leq q(x) \leq q_2 < +\infty,$$

and $q(x) < p^*(x)$ on $\partial\Omega$ with $p^*(x) = \begin{cases} \frac{(n-1)p(x)}{n-p(x)}, & \text{if } p_2 < n \\ +\infty, & \text{if } p_2 \geq n. \end{cases}$, then there is a continuous and compact embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\partial\Omega)$.

On Ω , we consider the following assumption.

(A) Ω is a bounded domain of $\mathbb{R}^2$ with smooth boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$, Γ_0 and Γ_1 are closed and disjoint, with $meas(\Gamma_0) > 0$, $\nu = (\nu_1, \nu_2)$ is the unit outward normal to $\partial\Omega$, $\eta = (-\nu_2, \nu_1)$ is the unit tangent positively oriented on $\partial\Omega$, and there is a fixed point $x_0 \in \mathbb{R}^2$ so that, for $w(x) = x - x_0$, $w \cdot \nu \leq 0$ on Γ_0 and $w \cdot \nu > 0$ on Γ_1 .

The existence and uniqueness of solutions to system (1.1) are provided in the following result, which can be established by the Faedo-Galerkin method as in the work of [17,26,30,32].

Proposition 2.1. Assume that (A) and (1.2)-(1.4) are satisfied and $(u_0, u_1) \in H_{\Gamma_0}^2(\Omega) \times L^2(\Omega)$, where $H_{\Gamma_0}^2(\Omega) = \{u \in H^2(\Omega) : u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_0\}$. Then, problem (1.1) has a unique global solution

$$u \in L^\infty((0, T); H_{\Gamma_0}^2(\Omega)), u_t \in L^\infty((0, T); L^2(\Omega)) \cap L^{m(\cdot)}((0, T) \times \Gamma_1).$$

As in [28], we define the following new variable

$$z(x, \rho, t) = u_t(x, t - \tau\rho), \quad \text{in } \Gamma_1 \times (0, 1) \times (0, \infty).$$

Then, system (1.1) is equivalent to

$$\left\{ \begin{array}{ll} u_{tt} + \Delta^2 u = 0, & \text{in } \Omega \times (0, \infty) \\ \tau z_t(x, \rho, t) + z_\rho(x, \rho, t) = 0, & \text{in } \Omega \times (0, 1) \times (0, \infty) \\ u = \frac{\partial u}{\partial \nu} = 0, & \text{on } \Gamma_0 \times (0, \infty) \\ \beta_1 u = 0, & \text{on } \Gamma_1 \times (0, \infty) \\ \beta_2 u = \alpha(t) |u_t|^{m(x)-2} u_t + \eta(t) |z(x, 1, t)|^{m(x)-2} z(x, 1, t), & \text{on } \Gamma_1 \times (0, \infty) \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & \text{in } \Omega \\ z(x, 0, t) = u_t(x, t), & \text{in } \Gamma_1 \times (0, \infty) \\ z(x, \rho, 0) = f(x, \tau \rho), & \text{in } \Gamma_1 \times (0, 1). \end{array} \right. \quad (2.1)$$

Now, let us define the bilinear form $a(.,.)$ as follows

$$a(u, v) = \int_{\Omega} \{u_{x_1 x_1} v_{x_1 x_1} + u_{x_2 x_2} v_{x_2 x_2} + \mu(u_{x_1 x_1} v_{x_2 x_2} + u_{x_2 x_2} v_{x_1 x_1}) + 2(1 - \mu)u_{x_1 x_2} v_{x_1 x_2}\} dx.$$

By Young's inequality, we find that

$$\begin{aligned} a(u, u) &= \int_{\Omega} \{u_{x_1 x_1}^2 + u_{x_2 x_2}^2 + 2\mu u_{x_1 x_1} u_{x_2 x_2} + 2(1 - \mu)u_{x_1 x_2}^2\} dx \\ &\leq \int_{\Omega} \left\{ u_{x_1 x_1}^2 + u_{x_2 x_2}^2 + 2\mu \left[\frac{1}{2} u_{x_1 x_1}^2 + \frac{1}{2} u_{x_2 x_2}^2 \right] + 2(1 - \mu)u_{x_1 x_2}^2 \right\} dx \\ &= \int_{\Omega} \{(1 + \mu) u_{x_1 x_1}^2 + (1 + \mu) u_{x_2 x_2}^2 + 2(1 - \mu)u_{x_1 x_2}^2\} dx \\ &\leq 2 \left(\|u_{x_1 x_1}\|^2 + \|u_{x_2 x_2}\|^2 + \|u_{x_1 x_2}\|^2 \right) \end{aligned}$$

and

$$\begin{aligned} a(u, u) &\geq \int_{\Omega} \left\{ u_{x_1 x_1}^2 + u_{x_2 x_2}^2 + 2\mu \left[-\frac{1}{2} u_{x_1 x_1}^2 - \frac{1}{2} u_{x_2 x_2}^2 \right] + 2(1 - \mu)u_{x_1 x_2}^2 \right\} dx \\ &= \int_{\Omega} \{(1 - \mu) u_{x_1 x_1}^2 + (1 - \mu) u_{x_2 x_2}^2 + 2(1 - \mu)u_{x_1 x_2}^2\} dx \\ &\geq (1 - \mu) \left(\|u_{x_1 x_1}\|^2 + \|u_{x_2 x_2}\|^2 + \|u_{x_1 x_2}\|^2 \right). \end{aligned}$$

Consequently, as $meas\Gamma_0 > 0$ and so Poincaré's inequality is applicable, we deduce that

$$\alpha_1 \|u\|_{H_{\Gamma_0}^2(\Omega)}^2 \leq a(u, u) \leq \alpha_2 \|u\|_{H_{\Gamma_0}^2(\Omega)}^2$$

for some positive constants α_1 and α_2 , which means that $\sqrt{a(u, u)}$ is an equivalent norm on $H_{\Gamma_0}^2(\Omega)$. At the end of this section, we introduce the energy functional

$$E(t) := \frac{1}{2} \int_{\Omega} |u_t|^2 dx + \frac{1}{2} a(u, u) + \tau \alpha(t) \int_{\Gamma_1} \int_0^1 \frac{\xi(x)}{m(x)} |z(x, \rho, t)|^{m(x)} d\rho d\Gamma$$

and

$$\xi(x) = \frac{1+q}{2} m(x) - q \quad (2.2)$$

where $0 < q < 1$ was given in (1.2).

3. Technical Lemmas

First, we state the following lemma which will be useful in what follows.

Lemma 3.1 ([19]). *Let u and z be functions in $H^4(\Omega)$ and $\mu \in \mathbb{R}$. Then we have*

$$\int_{\Omega} (\Delta^2 u) z dx = a(u, z) + \int_{\partial\Omega} \{(\beta_2 u) z - (\beta_1 u) \frac{\partial z}{\partial \nu}\} d\Gamma$$

and

$$\begin{aligned} \int_{\Omega} (w \cdot \nabla u) \Delta^2 u dx &= a(u, u) + \int_{\partial\Omega} \left[(\beta_2 u) (w \cdot \nabla u) - (\beta_1 u) \frac{\partial}{\partial \nu} (w \cdot \nabla u) \right] d\Gamma \\ &+ \frac{1}{2} \int_{\partial\Omega} w \cdot \nu [u_{x_1 x_1}^2 + u_{x_2 x_2}^2 + 2\mu u_{x_1 x_1} u_{x_2 x_2} + 2(1 - \mu) u_{x_1 x_2}^2] d\Gamma. \end{aligned}$$

Lemma 3.2. *The energy functional E satisfies, for some constant $c_0 > 0$,*

$$E'(t) \leq -c_0 \alpha(t) \int_{\Gamma_1} \left[|u_t|^{m(x)} + |z(x, 1, t)|^{m(x)} \right] d\Gamma. \quad (3.1)$$

Proof. By multiplying equation (2.1)₁ by u_t and integrating over Ω , using the boundary conditions, Lemma 3.1, and some manipulations, we obtain

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_{\Omega} (u_t^2 + a(u, u)) dx \\ &= -\alpha(t) \int_{\Gamma_1} |u_t|^{m(x)} d\Gamma - \eta(t) \int_{\Gamma_1} u_t z(x, 1, t) |z(x, 1, t)|^{m(x)-2} d\Gamma \end{aligned}$$

For the last term, we use Young's inequality with $p = m(x)$ and $p' = \frac{m(x)}{m(x)-1}$ to get

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_{\Omega} (u_t^2 + a(u, u)) dx \\ &\leq - \int_{\Gamma_1} \left(\alpha(t) - \frac{|\eta(t)|}{m(x)} \right) |u_t|^{m(x)} d\Gamma + |\eta(t)| \int_{\Gamma_1} \frac{m(x) - 1}{m(x)} |z(x, 1, t)|^{m(x)} d\Gamma. \end{aligned}$$

On the other hand, the use of (2.1)₂ and (1.2) gives

$$\begin{aligned} &\frac{d}{dt} \left[\tau \alpha(t) \int_{\Gamma_1} \int_0^1 \frac{\xi(x)}{m(x)} |z(x, \rho, t)|^{m(x)} d\rho d\Gamma \right] \\ &= \tau \alpha'(t) \int_{\Gamma_1} \int_0^1 \frac{\xi(x)}{m(x)} |z(x, \rho, t)|^{m(x)} d\rho d\Gamma \\ &\quad - \alpha(t) \int_{\Gamma_1} \int_0^1 \xi(x) z(x, \rho, t) |z(x, \rho, t)|^{m(x)-2} z_{\rho}(x, \rho, t) d\rho d\Gamma \\ &\leq -\alpha(t) \int_{\Gamma_1} \int_0^1 \frac{\xi(x)}{m(x)} \frac{\partial}{\partial \rho} |z(x, \rho, t)|^{m(x)} d\rho d\Gamma \\ &= \alpha(t) \int_{\Gamma_1} \frac{\xi(x)}{m(x)} |u_t|^{m(x)} d\Gamma - \alpha(t) \int_{\Gamma_1} \frac{\xi(x)}{m(x)} |z(x, 1, t)|^{m(x)} d\Gamma. \end{aligned}$$

Then, combining the above results and using (1.2) yield

$$E'(t) \leq -\alpha(t) \int_{\Gamma_1} \left[\left(1 - \frac{q + \xi(x)}{m(x)} \right) |u_t|^{m(x)} + \frac{\xi(x) - q(m(x) - 1)}{m(x)} |z(x, 1, t)|^{m(x)} \right] d\Gamma.$$

Using (2.2),

$$1 - \frac{q + \xi(x)}{m(x)} = \frac{1 - q}{2} = \frac{\xi(x) - q(m(x) - 1)}{m(x)}$$

Hence, (3.1) is established.

Let us mention that we will use c , throughout this paper, to denote a generic positive constant. In this section, we construct a Lyapunov functional $\mathcal{L}$ equivalent to E . For that, we first prove some lemmas.

Lemma 3.3. *The functional ψ defined by*

$$\psi(t) := \int_{\Omega} (w \cdot \nabla u) u_t dx$$

satisfies, along the solution of (1.1), the estimate

$$\psi'(t) \leq c \int_{\Gamma_1} u_t^2 d\Gamma - \int_{\Omega} u_t^2 dx - \frac{1}{2} a(u, u) + c\alpha(t) \int_{\Gamma_1} (|u_t|^{m(x)} + |z(x, 1, t)|^{m(x)}) d\Gamma. \quad (3.2)$$

Proof. Direct computations, using Lemma 3.1, we get

$$\begin{aligned} \psi'(t) &= \int_{\Omega} (w \cdot \nabla u_t) u_t dx + \int_{\Omega} (w \cdot \nabla u) u_{tt} dx \\ &= \frac{1}{2} \int_{\Omega} w \cdot \nabla u_t^2 dx - \int_{\Omega} (w \cdot \nabla u) \Delta^2 u dx \\ &= \frac{1}{2} \int_{\Omega} \left(\frac{1}{2} \nabla |w|^2 \right) \cdot \nabla u_t^2 dx - \int_{\Omega} (w \cdot \nabla u) \Delta^2 u dx \\ &= \frac{1}{2} \left[\int_{\partial\Omega} (w \cdot \nu) u_t^2 d\Gamma - \int_{\Omega} \left(\frac{1}{2} \Delta |w|^2 \right) u_t^2 dx \right] - \int_{\Omega} (w \cdot \nabla u) \Delta^2 u dx \\ &= \frac{1}{2} \left[\int_{\partial\Omega} (w \cdot \nu) u_t^2 d\Gamma - \int_{\Omega} (2) u_t^2 dx \right] - \int_{\Omega} (w \cdot \nabla u) \Delta^2 u dx \\ &= \frac{1}{2} \int_{\partial\Omega} (w \cdot \nu) u_t^2 d\Gamma - \int_{\Omega} u_t^2 dx - a(u, u) \\ &\quad - \int_{\partial\Omega} \left[(\beta_2 u) (w \cdot \nabla u) - (\beta_1 u) \frac{\partial}{\partial \nu} (w \cdot \nabla u) \right] d\Gamma \\ &\quad - \frac{1}{2} \int_{\partial\Omega} w \cdot \nu [u_{x_1 x_1}^2 + u_{x_2 x_2}^2 + 2\mu u_{x_1 x_1} u_{x_2 x_2} + 2(1 - \mu) u_{x_1 x_2}^2] d\Gamma \end{aligned} \quad (3.3)$$

Let us examine the integrals over Γ_0 in (3.3). As $u = 0$ on Γ_0 , then Γ_0 is a part of the level curve $u \equiv 0$ where ∇u is normal to this curve. Then, either $\nabla u = 0$ or ∇u is parallel to ν on Γ_0 . But, $\nabla u \cdot \nu = \frac{\partial u}{\partial \nu} = 0$ on Γ_0 confirms that $\nabla u = 0$ on Γ_0 . Hence, all the points on Γ_0 are critical points of u , and the discriminant $(u_{x_1 x_1} u_{x_2 x_2} - u_{x_1 x_2}^2)$ at all these points should be zero as $u = 0$ on Γ_0 . This means that

$$u_{x_1 x_1}^2 + u_{x_2 x_2}^2 + 2\mu u_{x_1 x_1} u_{x_2 x_2} + 2(1 - \mu) u_{x_1 x_2}^2 = (\Delta u)^2 \quad \text{on } \Gamma_0.$$

By similar arguments, one can show that $B_1 u = B_2 u = 0$ on Γ_0 and

$$\frac{\partial}{\partial \nu} (w \cdot \nabla u) = (w \cdot \nu) \Delta u \quad \text{on } \Gamma_0.$$

Therefore, with (2.1)_{4,5}, (3.3) becomes

$$\psi'(t) = \frac{1}{2} \int_{\Gamma_1} (w \cdot \nu) u_t^2 d\Gamma - \int_{\Omega} u_t^2 dx - a(u, u) + \frac{1}{2} \int_{\Gamma_0} (w \cdot \nu) (\Delta u)^2 d\Gamma$$

$$\begin{aligned}
& -\alpha(t) \int_{\Gamma_1} (w \cdot \nabla u) |u_t|^{m(x)-2} u_t d\Gamma - \eta(t) \int_{\Gamma_1} (w \cdot \nabla u) |z(x, 1, t)|^{m(x)-2} z(x, 1, t) d\Gamma \\
& - \frac{1}{2} \int_{\Gamma_1} w \cdot \nu [u_{x_1 x_1}^2 + u_{x_2 x_2}^2 + 2\mu u_{x_1 x_1} u_{x_2 x_2} + 2(1 - \mu) u_{x_1 x_2}^2] d\Gamma
\end{aligned}$$

and, as assumption (A) implies that there exists a constant R such that

$$|w(x)| \leq R \quad \text{for all } x \in \bar{\Omega}, \quad w \cdot v \leq 0 \quad \text{on } \Gamma_0, \quad w \cdot v > 0 \quad \text{on } \Gamma_1,$$

we obtain

$$\begin{aligned}
\psi'(t) & \leq \frac{R}{2} \int_{\Gamma_1} u_t^2 d\Gamma - \int_{\Omega} u_t^2 dx - a(u, u) \\
& -\alpha(t) \int_{\Gamma_1} (w \cdot \nabla u) |u_t|^{m(x)-2} u_t d\Gamma - \eta(t) \int_{\Gamma_1} (w \cdot \nabla u) |z(x, 1, t)|^{m(x)-2} z(x, 1, t) d\Gamma.
\end{aligned} \tag{3.4}$$

Now, we use Young's inequality, with $p(x) = m(x)$ and $p'(x) = \frac{m(x)}{m(x)-1}$ on $x \in \Gamma_1$, and the trace Theorem to get

$$\begin{aligned}
& -\alpha(t) \int_{\Gamma_1} (w \cdot \nabla u) |u_t|^{m(x)-2} u_t d\Gamma - \eta(t) \int_{\Gamma_1} (w \cdot \nabla u) |z(x, 1, t)|^{m(x)-2} z(x, 1, t) d\Gamma \\
& \leq \varepsilon \alpha(t) \int_{\Gamma_1} |w \cdot \nabla u|^{m(x)} d\Gamma + \alpha(t) \int_{\Gamma_1} C_\varepsilon(x) |u_t|^{m(x)} d\Gamma \\
& + \varepsilon |\eta(t)| \int_{\Gamma_1} |w \cdot \nabla u|^{m(x)} d\Gamma + |\eta(t)| \int_{\Gamma_1} C_\varepsilon(x) |z(x, 1, t)|^{m(x)} d\Gamma \\
& \leq \varepsilon(q+1)\alpha(t) \int_{\Gamma_1} |w \cdot \nabla u|^{m(x)} d\Gamma + \alpha(t) \int_{\Gamma_1} C_\varepsilon(x) \left(|u_t|^{m(x)} + q |z(x, 1, t)|^{m(x)} \right) d\Gamma \\
& \leq c\varepsilon \int_{\Gamma_1} \left(|w \cdot \nabla u|^2 + |w \cdot \nabla u|^{m_2} \right) d\Gamma + \alpha(t) \int_{\Gamma_1} C_\varepsilon(x) \left(|u_t|^{m(x)} + q |z(x, 1, t)|^{m(x)} \right) d\Gamma \\
& \leq c\varepsilon \left(a(u, u) + a(u, u)^{\frac{m_2}{2}} \right) + \alpha(t) \int_{\Gamma_1} C_\varepsilon(x) \left(|u_t|^{m(x)} + q |z(x, 1, t)|^{m(x)} \right) d\Gamma \\
& = c\varepsilon \left(1 + a(u, u)^{\frac{m_2-2}{2}} \right) a(u, u) + \alpha(t) \int_{\Gamma_1} C_\varepsilon(x) \left(|u_t|^{m(x)} + q |z(x, 1, t)|^{m(x)} \right) d\Gamma \\
& \leq c\varepsilon \left(1 + E(0)^{\frac{m_2-2}{2}} \right) a(u, u) + \alpha(t) \int_{\Gamma_1} C_\varepsilon(x) \left(|u_t|^{m(x)} + q |z(x, 1, t)|^{m(x)} \right) d\Gamma.
\end{aligned}$$

If we fix $\varepsilon = \frac{1}{2c \left(1 + E(0)^{\frac{m_2-2}{2}} \right)}$, then $m(x)$ is bounded and consequently $C_\varepsilon(x) = \frac{1}{p'(\varepsilon p)^{p'-1}} = \frac{\frac{m(x)-1}{m(x)}}{[m(x)]^{\frac{m(x)}{m(x)-1}} \varepsilon^{\frac{1}{m(x)-1}}}$ is bounded, and we get

$$\begin{aligned}
& -\alpha(t) \int_{\Gamma_1} (w \cdot \nabla u) |u_t|^{m(x)-2} u_t d\Gamma - \eta(t) \int_{\Gamma_1} (w \cdot \nabla u) |z(x, 1, t)|^{m(x)-2} z(x, 1, t) d\Gamma \\
& \leq \frac{1}{2} a(u, u) + c\alpha(t) \int_{\Gamma_1} \left(|u_t|^{m(x)} + |z(x, 1, t)|^{m(x)} \right) d\Gamma.
\end{aligned} \tag{3.5}$$

Combining the estimates (3.4) and (3.5), gives (3.2).

Lemma 3.4. *The functional*

$$K(t) := \tau \alpha(t) \int_{\Gamma_1} \int_0^1 e^{-\tau \rho} \xi(x) |z(x, \rho, t)|^{m(x)} d\rho d\Gamma$$

satisfies, for some positive constant γ_0 , the estimate

$$K'(t) \leq c\alpha(t) \int_{\Gamma_1} |u_t|^{m(x)} d\Gamma - \gamma_0\alpha(t) \left(\int_{\Gamma_1} \left[|z(x, 1, t)|^{m(x)} + \xi(x) \int_0^1 |z(x, \rho, t)|^{m(x)} d\rho \right] d\Gamma \right). \quad (3.6)$$

Proof. We use (1.2) and (2.1)₂ to find that

$$\begin{aligned} K'(t) &= \tau\alpha'(t) \int_{\Gamma_1} \int_0^1 e^{-\tau\rho} \xi(x) |z(x, \rho, t)|^{m(x)} d\rho d\Gamma \\ &\quad - \alpha(t) \int_{\Gamma_1} \int_0^1 e^{-\tau\rho} m(x) \xi(x) z(x, \rho, t) |z(x, \rho, t)|^{m(x)-2} z_\rho(x, \rho, t) d\rho d\Gamma \\ &\leq -\alpha(t) \int_{\Gamma_1} \xi(x) \left[\int_0^1 e^{-\tau\rho} \frac{\partial}{\partial \rho} |z(x, \rho, t)|^{m(x)} d\rho \right] d\Gamma \\ &= -\alpha(t) \int_{\Gamma_1} \xi(x) \left[e^{-\tau} |z(x, 1, t)|^{m(x)} - |z(x, 0, t)|^{m(x)} + \tau \int_0^1 e^{-\tau\rho} |z(x, \rho, t)|^{m(x)} d\rho \right] d\Gamma \\ &\leq \alpha(t) \int_{\Gamma_1} \xi(x) |u_t|^{m(x)} d\Gamma - \gamma_0\alpha(t) \left(\int_{\Gamma_1} \xi(x) \left[|z(x, 1, t)|^{m(x)} + \int_0^1 |z(x, \rho, t)|^{m(x)} d\rho \right] d\Gamma \right). \end{aligned}$$

As $\xi(x)$ is positive and bounded above and below, this gives (3.6).

Lemma 3.5. The functional $\mathcal{L}$ defined by

$$\mathcal{L}(t) := NE(t) + \psi(t) + K(t)$$

satisfies, for suitable choice of $N > 0$ and some $b_0 > 0$, that

$$\mathcal{L}'(t) \leq -b_0 E(t) + c \int_{\Gamma_1} u_t^2 d\Gamma \quad (3.7)$$

and

$$\mathcal{L}(t) \sim E(t). \quad (3.8)$$

Proof. From (3.1), (3.2) and (3.6), we obtain

$$\begin{aligned} \mathcal{L}'(t) &\leq c \int_{\Gamma_1} u_t^2 d\Gamma - \int_{\Omega} u_t^2 dx - \frac{1}{2} a(u, u) - (c_0 N - 2c) \alpha(t) \int_{\Gamma_1} |u_t|^{m(x)} d\Gamma \\ &\quad - (c_0 N + \gamma_0 - c) \alpha(t) \int_{\Gamma_1} |z(x, 1, t)|^{m(x)} d\Gamma - \gamma_0 \alpha(t) \int_{\Gamma_1} \int_0^1 \xi(x) |z(x, \rho, t)|^{m(x)} d\rho d\Gamma. \end{aligned}$$

Let us choose N large enough so that

$$c_0 N - 2c > 0.$$

Therefore,

$$\mathcal{L}'(t) \leq c \int_{\Gamma_1} u_t^2 d\Gamma - \int_{\Omega} u_t^2 dx - \frac{1}{2} a(u, u) - \gamma_0 \alpha(t) \int_{\Gamma_1} \int_0^1 \xi(x) |z(x, \rho, t)|^{m(x)} d\rho d\Gamma.$$

Since $\frac{\xi(x)}{m(x)} < \xi(x) \leq \frac{m_2 \xi(x)}{m(x)}$, then

$$\mathcal{L}'(t) \leq -b_0 E(t) + c \int_{\Gamma_1} u_t^2 d\Gamma$$

for some $b_0 > 0$. Now, by the aid of Young's and Poincaré's inequalities, one can easily deduce that

$$|\mathcal{L}(t) - NE(t)| \leq |\psi(t)| + |K(t)| \leq cE(t).$$

Hence, with N even larger (if needed), we can find some positive constants a_1, a_2 such that

$$a_1 E(t) \leq \mathcal{L}(t) \leq a_2 E(t)$$

which means that (3.8) is satisfied.

4. The Main Result

We are ready now to state and prove our main result.

Theorem 4.1. Assume that (1.2)-(1.4) and (A) hold. Then there exist positive constants k, k_1 and k_2 such that the solution of (1.1) satisfies, $\forall t \geq 0$,

$$E(t) \leq \begin{cases} ke^{-k_1 \int_0^t \alpha(s) ds}, & \text{if } m_2 = 2 \\ k_2 \left(1 + \int_0^t \alpha(s) ds\right)^{\frac{-2}{m_2-2}}, & \text{if } m_2 > 2 \end{cases}. \quad (4.1)$$

Proof. We multiply (3.7) by $\alpha(t)$ and get

$$\alpha(t)\mathcal{L}'(t) \leq -b_0\alpha(t)E(t) + c\alpha(t) \int_{\Gamma_1} u_t^2 d\Gamma. \quad (4.2)$$

By (3.1), if $m_2 = 2$, then $m(x) \equiv 2$ and

$$\alpha(t)\mathcal{L}'(t) \leq -b_0\alpha(t)E(t) - cE'(t).$$

Using (1.2),

$$(\alpha\mathcal{L} + cE)'(t) \leq -b_0\alpha(t)E(t).$$

Hence, by the fact that $\alpha\mathcal{L} + cE \sim E$, we easily obtain

$$E(t) \leq ke^{-k_1 \int_0^t \alpha(s) ds}$$

which gives (4.1)₁.

If $m_2 > 2$, we consider the following partition of Γ_1

$$\Gamma_* = \{x \in \Gamma_1 : |u_t| \geq 1\} \quad \text{and} \quad \Gamma_{**} = \{x \in \Gamma_1 : |u_t| < 1\}$$

and make use of Hölder's and Young's inequalities and (3.1) as follows:

$$\alpha \int_{\Gamma_*} u_t^2 d\Gamma \leq \alpha \int_{\Gamma_1} |u_t|^{m(x)} d\Gamma \leq -cE'(t), \quad (4.3)$$

$$\begin{aligned} \alpha \int_{\Gamma_{**}} u_t^2 d\Gamma &\leq c\alpha \left(\int_{\Gamma_{**}} |u_t|^{m_2} d\Gamma \right)^{\frac{2}{m_2}} \leq c\alpha \left(\int_{\Gamma_{**}} |u_t|^{m(x)} d\Gamma \right)^{\frac{2}{m_2}} \\ &\leq c\alpha^{1-\frac{2}{m_2}} \left(\alpha \int_{\Gamma_1} |u_t|^{m(x)} d\Gamma \right)^{\frac{2}{m_2}} = c\alpha^{1-\frac{2}{m_2}} (-E'(t))^{\frac{2}{m_2}} = \frac{cE^{\frac{m_2}{2}-1} \alpha^{1-\frac{2}{m_2}} (-E')^{\frac{2}{m_2}}}{E^{\frac{m_2}{2}-1}} \\ &\leq \frac{\delta \left[E^{\frac{m_2}{2}-1} \alpha^{1-\frac{2}{m_2}} \right]^{\frac{m_2}{m_2-2}} + C_\delta \left[(-E')^{\frac{2}{m_2}} \right]^{\frac{m_2}{2}}}{E^{\frac{m_2}{2}-1}} = \frac{\delta E^{\frac{m_2}{2}} \alpha - C_\delta E'}{E^{\frac{m_2}{2}-1}}. \end{aligned} \quad (4.4)$$

Inserting (4.3) and (4.4) into (4.2) and then multiplying by $E^{\frac{m_2}{2}-1}$ give

$$\alpha(t)E^{\frac{m_2}{2}-1}(t)\mathcal{L}'(t) \leq -\alpha(t)E^{\frac{m_2}{2}}(t) - cE'(t) + c\delta\alpha(t)E^{\frac{m_2}{2}}(t) - cC_\delta E'(t).$$

Consequently, as α and E are nonincreasing, picking δ small enough, we obtain, for some $b > 0$,

$$F'(t) \leq -b\alpha(t)E^{\frac{m_2}{2}}(t)$$

where $F = \alpha E^{\frac{m_2}{2}-1}\mathcal{L} + cE \sim E$. Hence we have, for some $b_1 > 0$,

$$F'(t) \leq -b_1\alpha(t)F^{\frac{m_2}{2}}(t)$$

$$\implies \int_0^t \frac{F'(s)}{F^{\frac{m_2}{2}}(s)} ds \leq -b_1 \int_0^t \alpha(s) ds$$

from which we easily deduce, for some constant $k_2 > 0$, that

$$E(t) \leq k_2 \left(1 + \int_0^t \alpha(s) ds \right)^{\frac{-2}{m_2-2}}$$

which gives (4.1)₂.

5. Examples and Remarks

Here, we give some examples and remarks to illustrate the energy decay rates obtained by Theorem 4.1.

Example 5.1. (A) If $m(x) \equiv 2$ and $\alpha(t) = (1+t)^{-\lambda}$ where $0 \leq \lambda \leq 1$, then

$$\begin{aligned} E(t) &\leq k e^{-\bar{k}(1+t)^{1-\lambda}}, & \text{if } 0 \leq \lambda < 1 \\ E(t) &\leq \frac{k}{(1+t)^{k_1}}, & \text{if } \lambda = 1. \end{aligned}$$

(B) If $m(x) \geq 2$, $m_2 = \sup_{x \in \Gamma_1} m(x) = 3$ and $\alpha(t) = (1+t)^{-\lambda}$ where $0 \leq \lambda \leq 1$, then

$$\begin{aligned} E(t) &\leq \frac{k_2}{(1+t)^{2(1-\lambda)}}, & \text{if } 0 \leq \lambda < 1 \\ E(t) &\leq \frac{k_2}{[1 + \ln(1+t)]^2}, & \text{if } \lambda = 1. \end{aligned}$$

Remark 5.2. In this paper, we investigated a plate system with boundary frictional damping and delay term having time-dependent coefficients $\alpha(t)$ and $\eta(t)$ and a variable exponent $m(x)$. Under the conditions $m(x) \geq 2$ and $|\eta(t)| \leq q\alpha(t)$, where $0 < q < 1$, we have derived explicit decay estimates for the energy functional associated with the system by applying the multiplier method. Our results demonstrate the effect of both α and m on the energy decay rates. These results contribute to the understanding of the dynamic behavior of such systems with variable exponent nonlinearities modulated by time dependent coefficients.

Remark 5.3. We should mention that the arguments used here need delicate modifications to treat with the case $1 < m(x) < 2$. Extending the analysis to cover this case would be of interest in our future work.

Data Availability Statement. My manuscript has no associated data.

Conflict of Interest Statement. This work does not have any conflicts of interest.

Acknowledgment. This work was supported by MASEP Research Group in the Research Institute of Sciences and Engineering at University of Sharjah.

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